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ABSTRACT

Crash severity is quantified by the change in velocity experienced by a vehicle during an impact along with the time duration over which that change in velocity occurs. Since the values of the input parameters for calculating the change in velocity are not known exactly, there is uncertainty associated with the calculated change in velocity. Accurate evaluation of the crash severity will, therefore, include analysis of the effect that uncertainties in the values of the input parameters have on the calculated change in velocity. Monte Carlo simulation, a statistical technique, enables the reconstructionist to evaluate the effect of uncertainty on the analysis of crash severity.

Use of the Monte Carlo simulation technique is beneficial since a reconstructionist can enter a range of values for each input parameter. A probability distribution can be assigned to the range of values, which indicates the likelihood that any value in that range corresponds to the actual value of the parameter. The simulation generates thousands of possible combinations of the input parameters selected from the specified ranges, monitors the results of the combinations and analyzes them statistically. Application of the Monte Carlo technique is intended to improve the legitimacy of crash severity analysis by helping the reconstructionist consider a wide range of possible solutions within the bounds of the imperfect data and report statistically meaningful ranges for the change in velocity.

This paper demonstrates the application of the Monte Carlo technique to impact severity analysis using a derived two-dimensional, rigid body, momentum-based impact model. Thorough guidance is given to aid the reconstructionist in integrating the momentum model with the Monte Carlo simulation technique and this method is illustrated with a case study. Since the impact model employs restitution constraints in the normal and tangential directions, the effect of uncertainty in formulating appropriate ranges for the values of the restitution coefficients is discussed.

NOTATION

m	mass
I	moment of inertia
V	velocity
ω	angular velocity
ΔV	change in velocity
$\Delta\omega$	change in angular velocity
P	impulse
a	X-coordinate of C.G.
b	Y-coordinate of C.G.
c	tangential coordinate of C.G.
d	normal coordinate of C.G.
e	coefficient of restitution
γ	orientation of the slip plane

Subscripts

1, 2	vehicle numbers
x	X-direction
y	Y-direction
t	tangential direction
n	normal direction
i	pre-impact
f	post-impact

INTRODUCTION

Crash severity, quantified by the change in velocity experienced by each vehicle during the impact[1], is a key parameter for assessing occupant injuries in motor vehicle accidents [10, 18, 25, 33]. The interest of litigators in knowing the potential for occupant injuries makes the accurate assessment of crash severity essential. However, the data necessary to evaluate the change in velocity experienced by each vehicle is not

1 Strictly speaking, crash severity is quantified by the change in velocity experienced by the vehicle during impact along with the time duration over which that change in velocity occurs [15]. In the discussion that follows, *crash severity* is used interchangeably with *change in velocity*. This is only to improve readability, and comes with the recognition that the time duration of the change in velocity is also an important parameter in assessing the crash severity.

known with complete certainty [5, 19, 20, 21, 24, 28, 29, 30, 31, 32]. The location of the point of impact and the points of rest, the orientation of the vehicle velocity vectors before and after impact, and the location where the resultant collision force is exchanged cannot be estimated or measured perfectly. Estimation of reasonable values for the coefficients of restitution at varying levels of crash severity and varying impact configurations is also fraught with uncertainty. Coefficients of friction are typically unmeasured and must be estimated. Even when they are measured, there will remain questions about the fidelity of the measurement in relation to the conditions at the time of the crash. This uncertainty associated with the input parameters results in subsequent uncertainty that is associated with the calculated change in velocity.

Monte Carlo simulation provides a statistical analysis technique to analyze the propagation of uncertainty from the input parameters to the final result, leading to statistically relevant conclusions regarding the probable ΔV experienced by a vehicle during an impact. The Monte Carlo technique allows a range of values to be specified for each input parameter of the impact model reflective of the level of certainty associated with that parameter. A probability distribution, that indicates the likelihood that any value in that range corresponds to the actual value, is attached to each of these ranges, and the simulation then generates possible combinations of these parameters based on the ranges and distributions. The results of these combinations are tracked and analyzed statistically. The final result is a probability distribution that expresses the likelihood that any value of the change in velocity corresponds to the actual value.

While application of Monte Carlo simulation to certain aspects of crash reconstruction has been addressed in the literature [19, 20, 30], previous papers have addressed trivial applications. This paper presents thorough and systematic guidance for the application of Monte Carlo simulation to the non-trivial case of impact analysis using a momentum-based rigid body impact model. This impact model employs restitution constraints in the normal and tangential directions, so the uncertainty inherent in identifying the normal and tangential directions is discussed. Guidance is given for the formulation of reasonable ranges for the coefficients of restitution.

ANALYTICAL UNCERTAINTY ANALYSIS

Analysis of crash severity should include analysis of the effect that uncertainty has on the calculated change in velocity. Several methods are available to analyze uncertainty in a dependent variable based on uncertainties in the independent variables. For the analysis in this paper, the dependant variable is the change in velocity and the independent variables consist of the coefficients of friction and restitution, post-impact travel distances, approach and departure angles, and vehicle masses.

The most common method of uncertainty analysis is to determine the upper and lower bounds of the dependent variable based on the lowest and highest possible values of each input parameter [5, 20]. This method has the advantage of being straightforward, even with non-linear equations. However, this upper and lower bound method does not result in information regarding the likelihood that any particular value within the obtained range for the change in velocity corresponds to the actual value. No basis is provided to conclude that crash severity values near the mean of the output range are any more likely to correspond to the actual value than those at the extremes. Further, the probability that the values of all of the independent variables fell at the extremes of these ranges in the actual crash is small. Therefore, the range generated for the dependent variable – the change in velocity – will be unrealistic and wider than necessary.

Another method of uncertainty analysis uses differential calculus to relate variations in the independent variables to the resulting variation in the dependent variable. Brach [5] and Tubergen [29] have expounded this method within the context of crash reconstruction and that discussion will not be repeated here. Suffice it to say that in the impact model described below, the linear and angular velocities are functions of ten independent variables, the values of which are not known with complete certainty. Analysis of the uncertainty using this method will, therefore, include calculating partial derivatives of the velocity equations with respect to each of these ten independent variables. Further, analysis of uncertainty by this method should be limited to cases where the variations are small, since the method approximates variations by linearizing the equations around some nominal value. And finally, as in the first method, this method gives no basis to draw statistical conclusions regarding individual values within the range of values obtained for the change in velocity.

An additional analytical method of uncertainty analysis accounts for the statistical nature of the ranges formulated for the independent parameters and allows for statistical conclusions to be drawn regarding the value of the dependent variable. A probability distribution that indicates the likelihood that any value in a range is likely to correspond to the actual value is specified for each independent variable. This probability distribution will be accounted for in analyzing the uncertainty associated with the final result. Statistical judgements can then be made regarding which values are most likely to occur. Brach [5] has detailed this method for a limited number of simple cases. However, this method becomes impractical for more complex sets of equations, such as the momentum equations considered below.

MONTE CARLO SIMULATION

Monte Carlo simulation provides a method of uncertainty analysis that produces statistical data similar to that produced by the analytical statistical method above, while avoiding the limitations of that method for complex equations. The Monte Carlo technique is an approximate method of considering the effect of variations in the independent variables on the uncertainty associated with the dependent variable. The Monte Carlo technique utilizes the power of a personal computer to accomplish thousands of repeated calculations of the dependent variable. These calculations are carried out with randomly selected values for the independent variables within the confines of ranges and probability distributions assigned to those ranges. The results are tracked and analyzed statistically. Conclusions relating to the likely change in velocity experienced by vehicle during a crash can then be drawn in a statistically legitimate manner.

The Monte Carlo analysis performed below uses a commercially available Monte Carlo simulator called **Crystal Ball**® that works in tandem with Microsoft® Excel. To perform the analysis, the equations for the impact model are configured in Excel. A range of values is then assigned to each independent parameter – called *assumptions* in **Crystal Ball**® – into which the actual value of that parameter should fall. Each range is then assigned a probability distribution that indicates the likelihood that any value in that range corresponds to the actual value of that parameter. Finally, the simulation runs a set number of calculations of the dependent variable, called the forecast in **Crystal Ball**®, with values for each parameter chosen randomly within the confines of the assigned ranges and probability distributions. Results of these calculations are monitored and the final result is a probability distribution that indicates the likelihood that any value in the resulting range corresponds to the actual value of the dependent variable (the change in velocity). Figure 1 is an example of the output probability distribution obtained for a vehicle's change in velocity from the case study below.

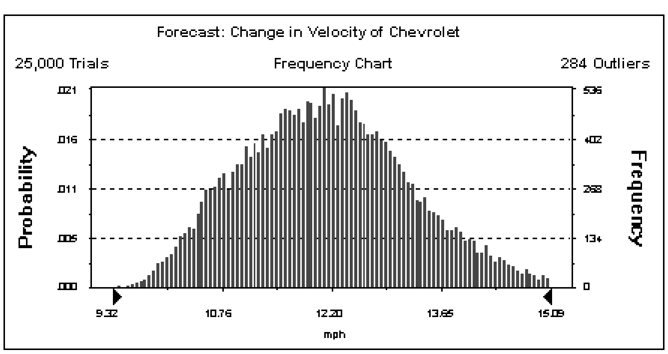

FIGURE 1

CHOOSING PROBABILITY DISTRIBUTIONS

When choosing probability distributions for each independent parameter, **Crystal Ball**® allows the user to select from a number of distributions including those depicted in Figure 2 below. Those distributions that might be applied within the context of crash reconstruction include the uniform, normal, and custom distributions.

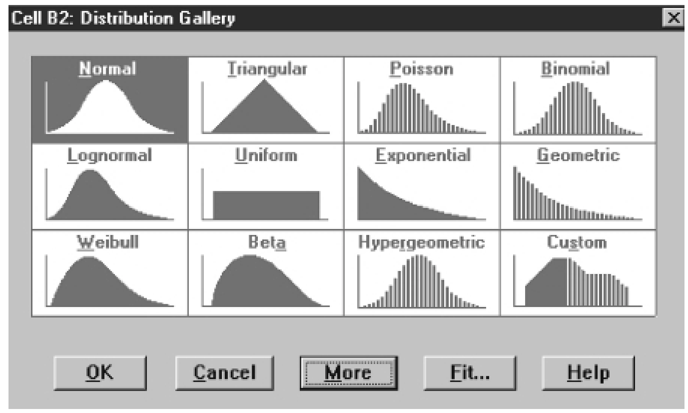

FIGURE 2

Selection of a uniform distribution implies that any value in the range is equally likely to occur. This distribution represents the most conservative assumption since the value of the parameter is allowed to vary more freely within the range than with any other distribution. The result is the widest possible variation in the output parameter [30]. The uniform distribution should be selected for most independent parameters in crash severity analysis since data that would justify choosing any other distribution is typically not available.

For instance, when a range and probability distribution is being assigned to the post-impact travel distance of a vehicle, this range will often be based on a single measurement of that distance. While we might expect random errors in measurement of the post-impact travel distance to vary normally, one measurement is insufficient to tell us the mean and standard deviation produced by those random errors. Further, uncertainty associated with the measurements of post-impact travel distances may go beyond random errors and include errors associated with identifying physical evidence. There may be uncertainty associated with the exact location where tire marks terminate, for example. Using a normal distribution with this independent variable would be unjustifiable with the provided data.

An exception to the use of a uniform distribution is for coefficients of friction. Goudie, et al presented data from 540 skid-to-stop tests under both wet and dry road conditions and found that random variations in the values of the coefficients of friction approximated a normal distribution [13]. Still, the use of a normal distribution to describe variations in the coefficient of friction for any particular reconstruction may be problematic. While Goudie tested three types of tires, his data is still for a single surface and a single vehicle. Random variations in the coefficient of friction might be expected to exhibit a normal distribution for any vehicle-tire-surface combination. However, each vehicle-tire-surface combination would have a unique mean value and a unique standard deviation. In any particular crash reconstruction, the mean value for that particular vehicle-tire-surface combination would not be known. Any number of normal distributions would be possible (Figure 3). Even extensive post-accident testing at the site would not necessarily yield the proper mean value and standard deviation since that testing would likely involve a different vehicle with different tires and suspensions characteristics. Furthermore, the generation of a statistically significant amount of data for a particular reconstruction is typically not feasible.

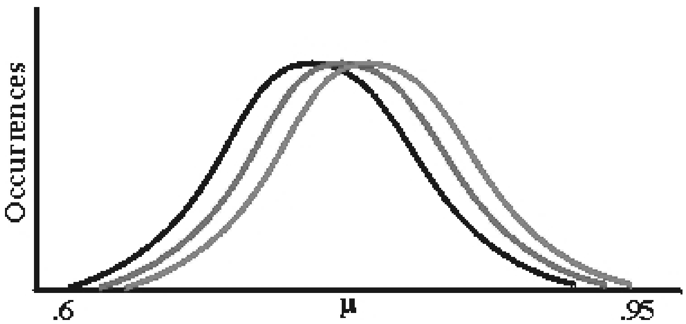

FIGURE 3

We could perhaps place bounds on the possible mean values and standard deviations and construct a custom distribution for variations in the coefficient of friction. For example, given some roadway surface, an experienced reconstructionist may be able to conclude that the mean value will likely occur between 0.7 and 0.85 (Figure 4). Using the custom distribution feature in **Crystal Ball**®, the range between 0.7 and 0.85 could be assigned a uniform distribution. The distribution could then tail off linearly on either side to the extremes of the ranges (Figure 4). This distribution displayed in Figure 4 by the dashed line is wide enough to encompass the breadth of values in the literature. Our intent is not to defend this range of values, but only the general shape of the distribution. A tighter range of friction coefficients could be specified if there is justification for that tighter range, such as consideration of the actual roadway surface involved in the crash.

This distribution shape in Figure 4 recognizes that for a given vehicle-tire-surface combination, variations in the coefficient of friction are likely to exhibit a normal distribution, but that in any particular reconstruction the mean value of that normal distribution would be unknown. At the same time, this distribution recognizes that limits can be placed on possible mean values, and values outside of these possible means need not be considered as likely as those within the range of possible means. This distribution also allows for a wide range of standard deviations.

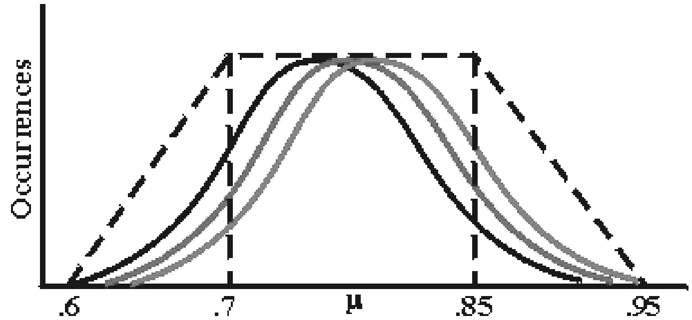

FIGURE 4

UNREALISTIC RESULTS

Some combinations of the independent variables generated by the Monte Carlo simulator may produce unrealistic results, and the output of the Monte Carlo simulation should be filtered to eliminate these combinations [30]. Mathematically, these unrealistic results occur because the equations of the impact model only constrain the output with respect to the independent variables that are actually contained in the equations of the impact model. For instance, there is no constraint in the equations of the impact model placed on the lateral acceleration that a vehicle can achieve. If a crash occurs while one vehicle is making a left turn, the analyst may be able to place limits on the speed of that vehicle based on the radius of the turn and the maximum lateral acceleration that the vehicle could achieve. The output data of the Monte Carlo simulation would have to be filtered based on this limit since the equations of the impact model contain no inherent constraint that would limit the speed of that vehicle based on maximum lateral acceleration.

The coefficients of restitution provide another criterion by which the result of each combination can be judged. The impact model formulated below does not require the restitution coefficients to be estimated in order to obtain a solution, and therefore the results are not constrainted with respect to reasonable restitution values. Output from the Monte Carlo simulation should be filtered to include only combinations that produce reasonable restitution values. The formulation of reasonable restitution coefficients is discussed below.

The most recent release of the **Crystal Ball**® software allows for automatic filtering of the results of the Monte Carlo simulation. The users manual should be consulted for specific instruction for accomplishing this filtering.

WHAT DO THE RESULTS MEAN?

Application of the Monte Carlo technique is intended to improve the legitimacy of crash severity analysis by enabling the reconstructionist to consider thousands of possible solutions within the bounds of the imperfect data that is available. Legitimacy is not inherent in application of the Monte Carlo technique, though, since the Monte Carlo technique cannot establish the fidelity of the impact model in any particular case. Meaningful application of the Monte Carlo technique must come after the assumptions of the impact model have been adequately satisfied.

For example, a typical momentum-based impact model assumes that tire forces can be neglected during the impact. If a model with this assumption is applied to analyze an impact of sufficient duration to make momentum losses from tire forces significant, the error in the solution induced by these momentum losses undermines the fidelity of the model for that accident. No amount of uncertainty analysis will solve this problem. A more complex impact model is required that will account for the tire forces that are neglected in the first model. Monte Carlo simulation could then be applied with the improved impact model.

Once the validity of the impact model has been established, the validity of the ranges and probability distributions assigned to the input parameters must likewise be established. In the same way that an improperly applied impact model undermines the effectiveness of the Monte Carlo technique, so do poorly chosen ranges for the input parameters. Test data from the literature should be used, the reliability of the available data should be assessed, and the range for each parameter should be sufficiently wide to encompass the full range of possible values. Once these conditions are met, then the results obtained from Monte Carlo simulation can be considered to give us meaningful information about the likely value of the change in velocity.

The results of the Monte Carlo analysis may be used in various ways depending on the issues relevant to the particular crash reconstruction. When the issue is crash severity, a 51-percentile range, indicating the range of values that are more probable than not, may be reported. When the impact speeds of the vehicles are also at issue, the 51-percentile range can likewise be reported [18]. If the overall distribution obtained straddles the speed limit, the reconstructionist may report the percent probability that the vehicle was exceeding the speed limit.

THE IMPACT MODEL

The two-dimensional impact model employed for demonstration of the Monte Carlo simulation technique makes use of the principle of impulse and momentum [6] and is derived with the following assumptions typical of momentum-based impact models [4, 7, 10, 16, 26, 27, 31, 32]:

1. **Tire forces and other external forces are assumed to be negligible compared to the collision forces.** This assumption allows for the principle of conservation of linear momentum to be applied and is generally considered valid for the analysis of vehicular crashes. However, instances exist when the duration of the impact is sufficiently long to make external forces significant. Fonda [11] has detailed a method for impact analysis that includes the momentum losses that result from external forces. Fonda’s method can be applied when the impact duration is too long to ignore external forces that cause significant momentum losses.

2. **The resultant impulse applied to each vehicle is concentrated at the impact center.** A number of papers have discussed the identification of the impact center [9, 14, 16, 31]. For the sake of brevity this discussion will not be expanded here. Suffice it to say that the impact center is “the point in the crush zone with a moment arm such that on the average the cross product of that moment arm with the impulse produces the correct rotational impulse” [31]. There is uncertainty associated with the identification of the impact center that should be accounted for in the analysis of crash severity. Further research should be done to explore the effect of variations in the location of the impact center on the calculated change in velocity.

3. **The interaction between the vehicles is assumed to occur instantaneously.** The collision forces are transferred instantaneously and the pre-impact and post-impact positions of the vehicles are assumed to coincide. This instantaneous transfer of the collision impulse is assumed to occur when the maximum crush to each vehicle has been reached. The validity of this assumption depends on the actual impact duration. In a real crash the force is not transferred instantaneously and movement of the vehicles does occur. The longer the impact duration, the more movement there will be and the less valid this assumption is.

4. **Each vehicle is treated as a rigid body with two translational degrees-of-freedom and one rotational degree-of-freedom.** Yaw rotations of the vehicles are considered, but roll and pitch rotations are neglected. Vertical motion is also neglected.

5. **Collision forces are treated as two-dimensional and assumed to act in the same plane in which the vehicles move.**

6. **The mass, center of gravity, and yaw moment of inertia for each vehicle are assumed unchanged by the impact.**

The impact model employs a fixed X-Y coordinate system located, for convenience, at the impact center, as shown in Figure 5.

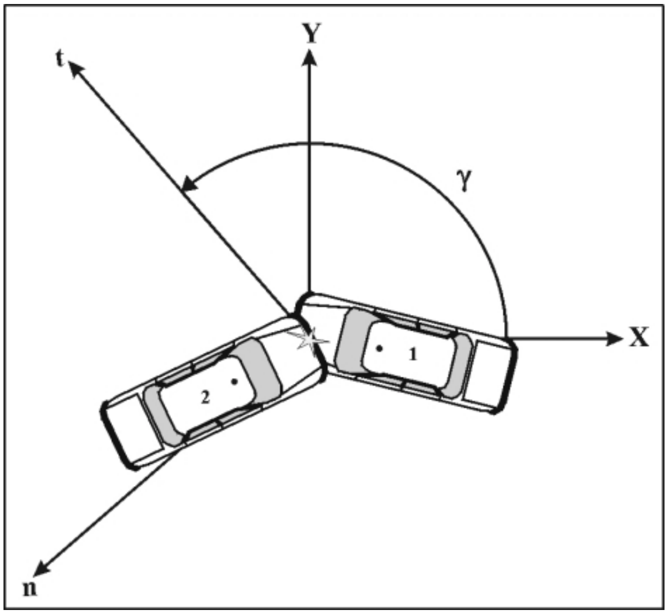

FIGURE 5

The location and orientation of this coordinate system are arbitrary. After the orientation of the fixed coordinate frame has been specified, the direction tangential to the impact surface is identified in relationship to the X-axis. The normal direction is by definition orthogonal to the tangential direction, or normal to the impact surface. The normal and tangential directions form a coordinate frame useful for setting up constraint equations. The orientation of the tangential direction (γ) along with the approach and departure angles of the vehicle velocities are measured counterclockwise from the positive X-axis.

Defining an arbitrarily oriented fixed coordinate system separate from the normal and tangential directions allows the orientation of the tangential direction to be included as an angular parameter in the impact analysis. This allows the uncertainty inherent in the identification of the normal and tangential directions to be considered in the analysis of crash severity.

Formulation of the impact equations begins with application of the principle of conservation of linear momentum [6] in the X and Y directions. The following equations are obtained:

$$m_1 V_{1xi} + m_2 V_{2xi} = m_1 V_{1xf} + m_2 V_{2xf} \tag{1}$$

$$m_1 V_{1yi} + m_2 V_{2yi} = m_1 V_{1yf} + m_2 V_{2yf} \tag{2}$$

The change in angular momentum of each vehicle is calculated by application of the principle of impulse and momentum. The following equations are obtained:

$$I_1(\omega_{1f} - \omega_{1i}) = -P_y a_1 + P_x b_1 \tag{3}$$

$$I_2(\omega_{2f} - \omega_{2i}) = -P_y a_2 + P_x b_2 \tag{4}$$

P_x and P_y are the components of the impulse applied to Vehicle #1 in the X and Y directions, respectively. An equal and opposite impulse is applied to Vehicle #2. Since the impulse applied to each vehicle is equal to the change in linear momentum experienced by that vehicle, we obtain the following equations:

$$P_x = m_1(V_{1xf} - V_{1xi}) = m_2(V_{2xi} - V_{2xf}) \tag{5}$$

$$P_y = m_1(V_{1yf} - V_{1yi}) = m_2(V_{2yi} - V_{2yf}) \tag{6}$$

Substitution of equations (5) and (6) into (3) and (4) yields four equations describing the impact. However, a general two-dimensional impact model includes 3 degrees of freedom for each vehicle, therefore, six equations are required to completely describe the impact. Two constraint equations relating the components of the closing speed to the components of the separation speed at the impact center provide the additional two equations necessary for a complete impact model.

This impact model is similar to that derived by Ishikawa [16] in that restitution constraints in the normal and tangential directions provide the two additional equations necessary for a complete model. The restitution equations are as follows:

$$e_n = \frac{V_{sep,n}}{V_{close,n}} = \frac{V_{2nf} - c_2\omega_{2f} - V_{1nf} + c_1\omega_{1f}}{V_{1ni} - c_1\omega_{1i} - V_{2ni} + c_2\omega_{2i}} \tag{7}$$

$$e_t = \frac{V_{sep,t}}{V_{close,t}} = \frac{V_{2tf} - d_2\omega_{2f} - V_{1tf} + d_1\omega_{1f}}{V_{1ti} - d_1\omega_{1i} - V_{2ti} + d_2\omega_{2i}} \tag{8}$$

where c_1 and c_2 are the tangential coordinates of the centers of gravity of vehicles #1 and #2, respectively, and d_1 and d_2 are the normal coordinates of the centers of gravity of vehicles #1 and #2, respectively. The numerators of equations (7) and (8) are the normal and tangential components of the separation speed. The denominators are the normal and tangential components of the closing speed. Transformation of restitution equations (7) and (8) to the X-Y frame result in the following equations:

$$e_n = \frac{(V_{sep,x})\sin\gamma + (V_{sep,y})\cos\gamma}{(V_{close,x})\sin\gamma + (V_{close,y})\cos\gamma} \tag{9}$$

$$e_t = \frac{(-V_{sep,x})\cos\gamma + (V_{sep,y})\sin\gamma}{(-V_{close,x})\cos\gamma + (V_{close,y})\sin\gamma} \tag{10}$$

where,

$$V_{sep,x} = (V_{1xf} + b_1\omega_{1f} - V_{2xf} - b_2\omega_{2f}) \tag{11}$$

$$V_{sep,y} = (V_{2yf} - a_2\omega_{2f} - V_{1yf} + a_1\omega_{1f}) \tag{12}$$

$$V_{close,x} = (V_{2xi} + b_2\omega_{2i} - V_{1xi} - b_1\omega_{1i}) \tag{13}$$

$$V_{close,y} = (V_{1yi} - a_1\omega_{1i} - V_{2yi} + a_2\omega_{2i}) \tag{14}$$

In these equations a_1 and a_2 are the X-coordinates of the centers of gravity of vehicles #1 and #2, respectively, and b_1 and b_2 are the Y-coordinates of the centers of gravity of vehicles #1 and #2, respectively. Equations (1) through (4), in conjunction with equations (5) and (6), and equations (9) and (10) now provide a complete description of the impact.

The values of the normal and tangential restitution coefficients are allowed to vary between values of -1 and 1. Ishikawa introduced the idea of negative restitution within the context of accident reconstruction [16]. The result of allowing restitution to fall below 0 is that the impact model does not require a common velocity to be achieved during the impact. This is advantageous since the impact model can be used to analyze sideswipe or break-through collisions where a common velocity is never achieved. However, once the values of the restitution coefficients drop below 0, the classical definition of the restitution constraints as a ratio of deformation and restitution impulses has been abandoned.

If the orientation of the vehicle velocities before and after impact can be estimated, the number of unknowns is reduced to four and equations (1) through (4), with (5) and (6), are sufficient to produce a solution. Writing equations (1) through (4) with approach and departure angles yields equations (15) through (18).

$$V_{1i} = \frac{m_1 V_{1f}\sin(\beta_1 - \alpha_2) - m_2 V_{2f}\sin(\alpha_2 - \beta_2)}{m_1\sin(\alpha_1 - \alpha_2)} \tag{15}$$

$$V_{2i} = \frac{1}{m_2\cos\alpha_2}\left(-m_1 V_{1i}\cos\alpha_1 + m_1 V_{1f}\cos\beta_1 + m_2 V_{2f}\cos\beta_2\right) \tag{16}$$

$$\omega_{1f} = \frac{m_1}{I_1}V_{1f}(\cos\beta_1 b_1 - \sin\beta_1 a_1) - \frac{m_1}{I_1}V_{1i}(\cos\alpha_1 b_1 - \sin\alpha_1 a_1) + \omega_{1i} \tag{17}$$

$$\omega_{2f} = \frac{m_2}{I_2}V_{2f}(\cos\beta_2 b_2 - \sin\beta_2 a_2) - \frac{m_2}{I_2}V_{2i}(\cos\alpha_2 b_2 - \sin\alpha_2 a_2) + \omega_{2i} \tag{18}$$

Application of the impact model demonstrated below assumes that approach and departure angles can be estimated and equations (15) through (18) are utilized in the Excel spreadsheet for the Monte Carlo analysis. The coefficients of restitution become unnecessary to generate a solution, but they are used to monitor possible solutions for reasonableness. The issue of reasonable and unreasonable solutions is discussed below.

Finally, equations for the change in velocity experienced by each vehicle during the impact are obtained by vector subtraction of the final velocities from the initial velocities.

$$\Delta V_j = \sqrt{\Delta V_{jx}^{\ 2} + \Delta V_{jy}^{\ 2}} \tag{19}$$

where,

$$\Delta V_{jx} = V_{jf}\cos\beta_j - V_{ji}\cos\alpha_j \tag{20}$$

$$\Delta V_{jy} = V_{jf}\sin\beta_j - V_{ji}\sin\alpha_j \tag{21}$$

$$j = 1,2 \tag{22}$$

The principle direction of force (PDOF) is calculated geometrically by the following equation:

$$PDOF_{\Delta Vj} = 180 + \left(\tan^{-1}\frac{\Delta V_{jy}}{\Delta V_{jx}}\right) - \alpha_j \tag{23}$$

RESTITUTION – THEORETICAL ASPECTS

Traditionally, the constraint equations for two-dimensional rigid body impact models have been formulated by defining a coefficient of restitution in the normal direction and an equivalent friction coefficient in the tangential direction [4]. In classical mechanics, the normal coefficient of restitution is defined at the impact center as the ratio of the normal impulses during the restitution and deformation phases [6].

$$e_n = \frac{normal\ restitution\ impulse}{normal\ deformation\ impulse} = \frac{\int F_{Rn}dt}{\int F_{Dn}dt} \tag{24}$$

Since the impulses during these phases are equal to the change in momentum of each vehicle during that phase, substitution, algebraic manipulation, and consideration of the two-body system renders the familiar form of the normal coefficient of restitution [6].

$$e_n = \frac{V_{2nf} - V_{1nf}}{V_{2ni} - V_{2ni}} \tag{25}$$

Definition of the coefficient of restitution as the ratio of the restitution and deformation impulses requires that the vehicles reach a common velocity in the normal direction at the impact center at some time during the collision. The dividing line between the period of deformation and the period of restitution is by definition the instant at which this common velocity is achieved and the restitution coefficient indicates the extent to which the vehicles rebound from this common velocity. This common velocity assumption constrains the value of the coefficient of restitution to lie between 0 and 1.

The equivalent coefficient of friction has been defined by Brach [5] as follows:

$$\mu = \frac{P_t}{P_n} \tag{26}$$

Thus, the equivalent friction coefficient is a ratio that indicates the magnitude of the tangential impulse relative to the normal impulse. As Brach has noted, this coefficient accounts not only for sliding friction between the vehicles, but also for structural engagement. This parameter controls the orientation of the principle direction of force relative to the crush surface.

The impact model derived above parts with the equivalent coefficient of friction as the constraint parameter in the tangential direction. Instead, coefficients of restitution are defined in both the normal and tangential directions. These are not coefficients of restitution in the traditional sense, though, since they are allowed to vary between –1 and 1. This has the effect of relaxing the common velocity requirement since a restitution value of less than 0 implies that a common velocity was never achieved. If this allowance for negative restitution values is made, the coefficient of restitution can no longer be defined as the ratio of the restitution and deformation impulses since there is no longer a clear dividing line separating the impact into two phases. Instead, the normal and tangential restitution constraints are defined on the pattern of equation (24) without direct reference to the impulses.

Parting with the traditional impulse ratio definition of the coefficients of restitution and the common velocity assumption that accompanies that definition is advantageous within the context of crash reconstruction since the impact model can then handle collisions where a common velocity is not reached at any time during the collision. Sideswipe collisions are the most prominent example. Another is crashes where one vehicle engages the corner of the other and simply breaks through the structure of that vehicle [16].

RESTITUTION – APPROPRIATE RANGES

The model derived above assumes that the approach and departure angles of the vehicle velocities can be estimated and so the coefficients of restitution are not necessary to generate solutions. However, potential solutions must be checked for reasonable restitution values. Thus ranges for the coefficients of restitution must be estimated even though the model does not require them. Rejection of unreasonable solutions is discussed below.

Formulation of ranges for the coefficients of restitution can be problematic since restitution in vehicular collisions is a complex structural phenomenon where the non-homogeneity of the vehicle structure causes the restitution response to vary depending on what portions of the vehicle are engaged in the impact. Also, each vehicle combination will exhibit a unique collision response at any given impact configuration and severity [1, 14]. Estimation of the coefficients of restitution for a particular crash will involve applying staged collision data that does not strictly represent the specific vehicle structures, the impact configuration, or the severity of that particular crash.

NORMAL RESTITUTION

<u>Crash Severity and Normal Restitution</u> - Researchers have generally concluded that normal restitution values decrease as crash severity increases [1, 14, 16, 17, 25]. Some impact models have exploited this trend by assuming that restitution approaches zero for high severity collisions and can therefore be neglected in these collisions [21]. As the severity of a collision decreases, however, the effect of restitution on the change in velocity experienced by each vehicle becomes more significant, and at some point restitution cannot be neglected. Even in high severity collisions, consideration of the restitution response can improve the fidelity of the impact model and may be important with certain impact configurations. Restitution values that approach zero in high severity collisions are best established for frontal barrier impacts, and more testing and discussion is needed before concluding that this assumption is valid for all impact geometries.

The non-homogeneity of the vehicle structure introduces important exceptions to the trend of decreasing restitution with increasing crash severity. Frontal barrier impact (VTB) tests and front-to-rear vehicle-to-vehicle (VTV) collisions are the most represented collision types in the literature and, therefore, discontinuities in restitution response are best documented for the front structure of the vehicle. The front structure of a vehicle can be considered as several generally homogeneous regions, each with elastic and plastic deformation regions. As crush progresses through these regions, a new region of elastic deformation will follow a region of plastic deformation. When the crush enters a new elastic deformation region, the result is a localized discontinuity in the relationship between crash severity and restitution [14].

Siegmund, Bailey, and King [25] conducted 660 low-speed impact tests and found a slight rise in the coefficient of restitution for some vehicles in frontal barrier and front-to-rear VTV impacts that corresponded to the bottoming out of the isolators. Monson and Germane [23] analyzed 181 NHTSA full width VTB impacts and reported an average coefficient of restitution that was higher at 35 mph than at 30 mph for vehicles with transverse-mounted engines. This rise in restitution was attributed to engagement between the engine and the cowl panel. In both instances, rises in restitution with increasing crash severity corresponded to the entry of the deformation into a new elastic region.

Another fundamental problem with using crash severity to estimate restitution is that the crash severity is the very parameter being calculated. If the impact severity for a particular crash has not been determined, then the impact severity cannot be strictly correlated to the restitution. Estimates of restitution based on crash severity must inevitably be based on visual inspection of the damage to the vehicles. Such estimates may be legitimate based on the experience of the reconstructionist, but they remain estimates with some level of uncertainty involved.

Impact Configuration and Restitution - Ishikawa [17] has presented analysis of a series of 45 staged vehicle-to-vehicle collisions at a number of impact configurations and severities. Thirty-two of these collisions were side impacts to the target vehicle, and the remaining 13 were frontal impacts to the target vehicle. For the side impacts, impact angles included 75 degrees, 90 degrees, 120 degrees, 135 degrees, and 150 degrees. For the frontal impacts, impact angles included 120 degrees, 135 degrees, 150 degrees, 165 degrees, and 180 degrees. The cars used in the collisions were Japanese passenger cars weighing approximately 1000 kg.

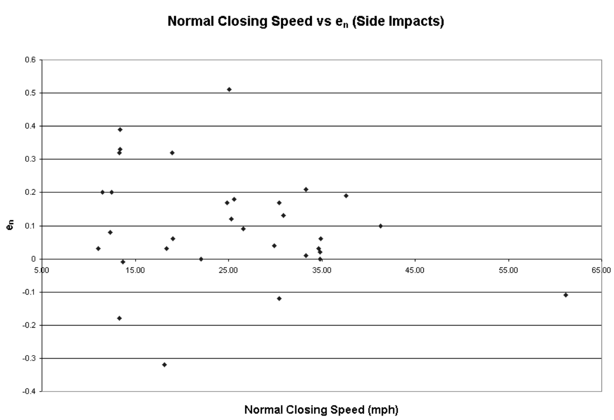

FIGURE 6

Ishikawa's data seems to indicate that any relationship between crash severity and normal restitution breaks down unless collisions are first grouped by their impact configuration. Figure 6 depicts normal restitution values reported by Ishikawa for 32 side impact tests plotted against the closing speed in the normal direction. The impact configurations and severities vary widely in the data, and the results do not show a clear correlation between crash severity and normal restitution for this eclectic mix of collisions.

Figure 7 depicts normal restitution values reported by Ishikawa for 13 frontal impact tests plotted against the closing speed in the normal direction. Again, the impact configurations and severities vary widely and again, there is no clear correlation between crash severity and normal restitution.

More research should be done to break Ishikawa's data down into classes by impact configuration to see if the trend of decreasing normal restitution with increasing crash severity holds within these groups. There may not be sufficient data at this time to demonstrate this conclusively, but a conceptual framework might be laid to direct the expansion of the available staged collision data.

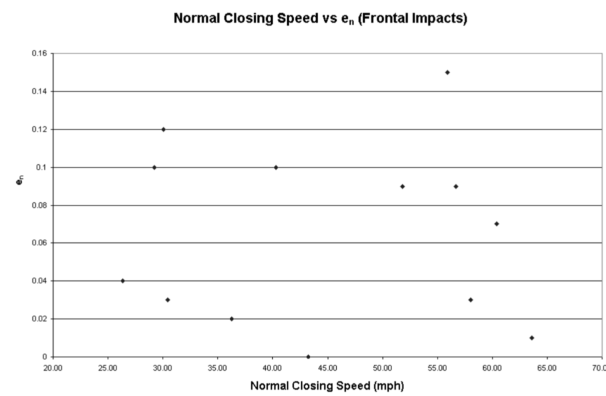

FIGURE 7

While we may expect to find decreasing restitution values with increasing crash severity for collisions grouped by their impact configuration, there may be certain impact configurations that do not reach the low restitution values observed in high severity frontal collisions. Side impacts at the axle may be one of these classes. Impact configuration is, therefore, an important consideration when estimating restitution. More experimental and theoretical work needs to be completed to clarify restitution's dependence on collision geometry.

Variations in the Data - Normal restitution values show considerable variation from test to test. For example, barrier tests at 30 mph analyzed by Monson and Germane exhibited coefficients of restitution varying between approximately 0.04 and 0.24 with one test falling outside this range at about 0.35. Tests at 35 mph exhibited coefficients of restitution varying between 0.08 and 0.23. Scatter in coefficients of restitution observed during their low-speed VTB and VTV impact tests led King, Siegmund, and Bailey to conclude that "accurate estimates of the coefficients of restitution for a given collision can only be determined by performing tests with the actual vehicles" [18]. Their tests exhibited restitution values between 0.25 and 0.75 and they found "noticeable differences between the different bumper types."

TANGENTIAL RESTITUTION

Estimating values for the coefficient of restitution in the tangential direction proves more problematic than estimating corresponding values in the normal directions. Data from staged collisions is limited. The primary data available for estimating coefficients of restitution in the tangential direction (and equivalent friction coefficients) again comes from Ishikawa's staged collision data [16, 17].

Side Impact Data - Figure 8 depicts tangential restitution values reported by Ishikawa for 32 side impact tests plotted against tangential closing speed.

This group of data shows the greatest potential for displaying a general relationship between crash severity and restitution. There is a trend of decreasing tangential restitution with increasing tangential closing speed. This trend makes intuitive sense in light of the physical interpretation of the tangential coefficient of restitution. A negative restitution coefficient implies that a common velocity was never reached in the tangential direction. We would expect that the higher the closing speed in the tangential direction, the harder it would become for structural engagement to cause a common velocity to be reached. Thus, in general, at lower closing speeds the engagement of the vehicle structures seem more capable of bringing the vehicles to a common velocity in the tangential direction.

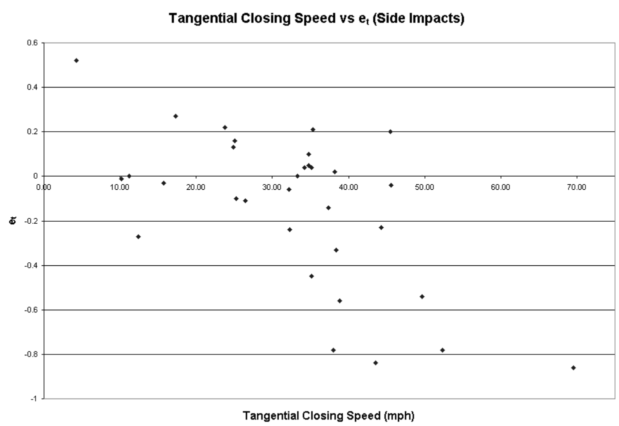

FIGURE 8

Still, there is considerable variance in the relationship between tangential closing speed and tangential restitution, indicating that there are other important factors. Intuitively, we would expect the normal closing speed to affect tangential restitution values, as well, since normal closing speed effects the extent of structural engagement. A higher closing speed in the normal direction would generally yield more significant structural engagement and therefore a greater ability by the vehicle structure to produce a common velocity in the tangential direction.

This intuition is confirmed by Ishikawa's staged collision data. Figure 9 isolates the side impact tests with tangential closing speeds between 30 and 40 mph and plots their tangential restitution values against the normal closing speed.

The structural engagement between the vehicles appears to remain superficial up to around 15 mph, with the normal closing speed playing little role in the tangential restitution value. Above a 15 mph normal closing speed there appears to be a general trend of increasing tangential restitution values with increasing normal closing speed.

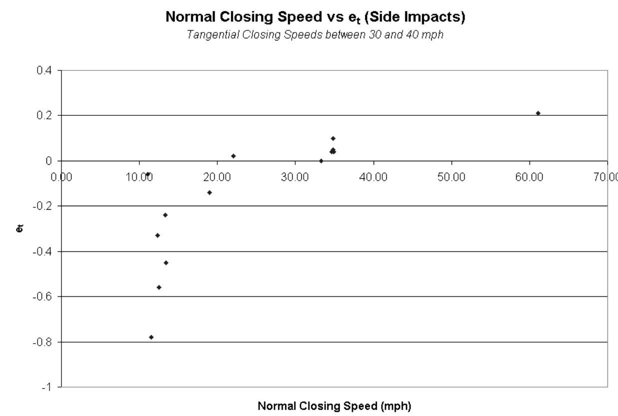

FIGURE 9

Frontal Impact Data - Figure 10 depicts tangential restitution values reported by Ishikawa for 13 frontal impact tests plotted against the closing speed in the tangential direction. Again, there is no clear correlation between the tangential restitution values and the crash severity.

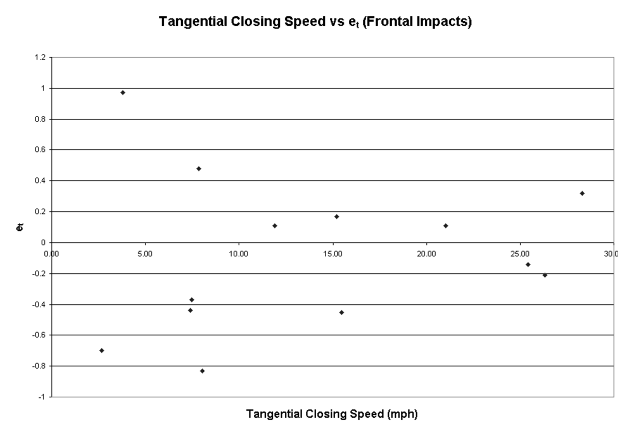

FIGURE 10

To estimate a reasonable range of tangential restitution values for a particular crash, the reconstructionist should start by establishing a correlation between that particular crash and the physical interpretation of the tangential restitution coefficient. For instance, for many sideswipe collisions a common velocity is never reached and therefore the reconstructionist should expect a negative restitution value. Likewise, the reconstructionist may be able to conclude that structural engagement between the vehicles was such that a common velocity in the tangential direction was reached and that the tangential coefficient of restitution should be zero or greater.

Beyond that general characterization, the reconstructionist should consider the tangential restitution values from collisions that most resemble the crash in question. This data is limited and needs expanding. Ishikawa's data is helpful since he reports values for both e_t and μ. Other researchers report neither.

GENERAL CONSIDERATIONS

Selection of Normal and Tangential Directions - Restitution values reported for staged VTV tests, such as those reported by Ishikawa, depend on the reconstructionists selection of the normal and tangential directions. Since there is uncertainty associated with the selection of these directions, analysis of the same data by another reconstructionist would likely lead to different restitution values. It would be worthwhile to investigate the effect of the uncertainty in the normal and tangential directions. At any rate, reconstructionists should recognize that restitution values published in the literature have uncertainty associated with them.

Summary of Available Data - Reconstructionists should be familiar with the staged collision data that is available and use this data for estimating coefficients of restitution. References 2, 3, 8, 15, 16, 17, 21, and 23 are excellent sources of data. In the analysis of any particular crash the reconstructionist should use the data from stage collisions that best mimics that crash. The available data will rarely be strictly analogous and the reconstructionist should formulate ranges for the restitution values that reflect the level of uncertainty associated with those values. The ranges should be wide enough to encompass the full range of possible restitution values.

For the foreseeable future, frontal barrier impact data will continue to be the most readily available. While the number of staged VTV collisions in the literature at varying impact configurations and severities continues to grow, there remain large gaps in the data for many impact configurations and for most vehicle combinations. Even if these gaps were filled for impact configuration, testing of every vehicle combination is not feasible and this gap is permanent. Analysis of crash severity should give an accurate picture of the uncertainty inherent in estimates of the coefficients of restitution.

CASE STUDY

BACKGROUND - The following case study is an attempt to tie together the ideas that have been discussed, including (1) the application of the impact model, (2) the formulation of ranges and probability distributions for the independent parameters, (3) the filtering of the data for realistic restitution values, and (4) the formulation of conclusions based on the filtered data. Detailed instructions for the use of **Crystal Ball**® software are contained in the user manual and are not repeated here.

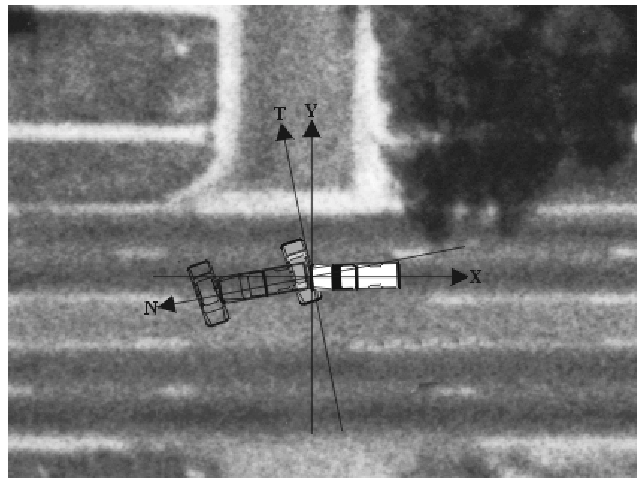

FIGURE 11

The crash considered is an intersection collision involving a Chevrolet pickup and a Nissan sedan. The crash occurred when the Nissan attempted to make a left turn in front of the oncoming Chevrolet. Figure 11 depicts the collision geometry, the point of impact, the rest positions of the vehicles, and the coordinate system used for application of the momentum model. Both vehicles traveled approximately 22 feet after impact.

Parameter	Low End	High End
Weight of Chevy	6661	6861
Chevy Moment of Inertia	4712	5208
Chevy Approach Angle	181	191
Chevy Departure Angle	187	197
Chevy Post-Impact Travel Distance	20	22
Post-Impact Deceleration	0.60	0.95
Weight of Nissan	3410	3510
Nissan Moment of Inertia	1849	2043
Nissan Approach Angle	272	292
Nissan Departure Angle	192	202
Nissan Post-Impact Travel Distance	20	24
Post-Impact Deceleration	0.6	0.95
Chevy X-Coordinate of Impact Center	97.5	103.5
Chevy Y-Coordinate of Impact Center	-16	-10
Nissan X-Coordinate of Impact Center	-3.2	3.2
Nissan Y-Coordinate of Impact Center	-28.6	-22.6
Orientation of Tangential Direction	96	106

TABLE 1

MONTE CARLO SIMULATION - The equations of the impact model were entered in an Excel spreadsheet. Each of the independent parameters was assigned a range of values and a uniform probability distribution. The range used for each variable is listed in Table 1.

A few of these ranges deserve comment. The Chevy truck was carrying a load, but details regarding the weight of the load were not available. Thus, the weight of the Chevrolet was allowed to vary more widely than the weight of the Nissan. Occupant weights were reasonably well known. Vehicle weights were taken from published data.

The point of impact and the points of rest were documented by the police and were considered well established. The motion of both vehicles between the point of impact and the points of rest was relatively straight. Therefore, the departure angles of both vehicles were well established and were only allowed to vary within a 10-degree range.

The approach angle of the Chevy was known reasonably well. There were no pre-impact skid marks by the Chevy before impact and the driver indicated that he had very little time to react before the accident. This was consistent with time-space analysis performed by these engineers. The driver of the Chevy may have been able to steer before impact, so the approach angle was allowed to vary between 181 and 191 to allow for the possibility of an evasive steer to the left before impact.

The ranges on the post-impact deceleration rates of the vehicles are wide enough to encompass the breadth of values in the literature for dry asphalt. We do not intend to prescribe the use of this range. The reconstructionist should use data from the literature and knowledge of the specific surface and formulate a range on a case by case basis.

Finally, the impact center was located using the crush energy analysis portion of EDCRASH. Here, again, we are not prescribing this as the preferred method for locating the impact center.

Having assigned these ranges and a uniform distribution to each of the independent variables, the Monte Carlo simulation was run with 200,000 trials. The data generated by the simulation was filtered to eliminate unrealistic results based on conceptual considerations of the restitution. Since there was significant engagement between the vehicles, the authors concluded that the vehicles reached a common velocity in both the normal and tangential directions. The restitution values in both the normal and tangential directions, therefore, had to be greater than or equal to zero. Restitution values were also required to be less than 1, since these values would imply that energy was added to the collision.

Tangential restitution values were further restricted to a range of 0.0 to 0.25 since it was clear from physical evidence that the vehicles did not depart far from the common velocity reached in the tangential direction. Finally, impact speeds calculated for the Nissan were required to be positive. Of the original 200,000 combinations of the parameters, 3,481 or approximately 2% of these combinations survived these rejection criteria. This is similar to the percentage of accepted values reported by other authors [30]. However, in the experience of the authors, this percentage varies widely from case to case.

Histograms were generated based on these remaining values and are displayed in Figures 12 through 15. Table 3 lists the mean, standard deviation, and 51% range for these histograms.

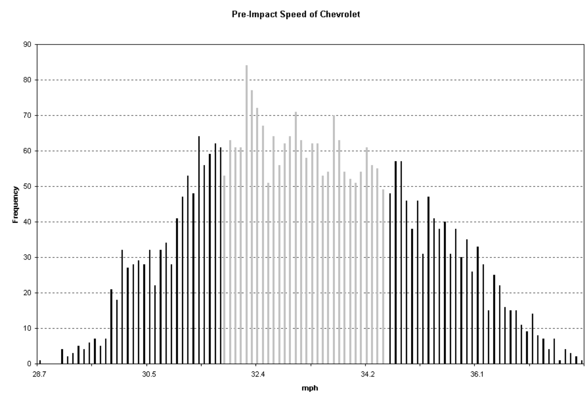

FIGURE 12

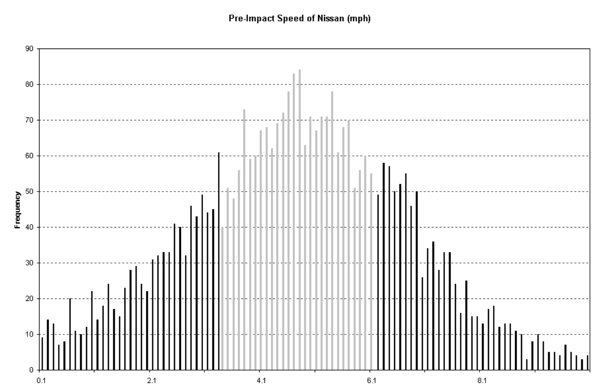

FIGURE 13

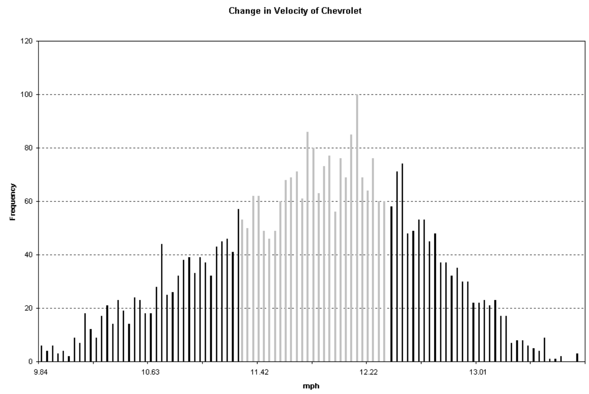

FIGURE 14

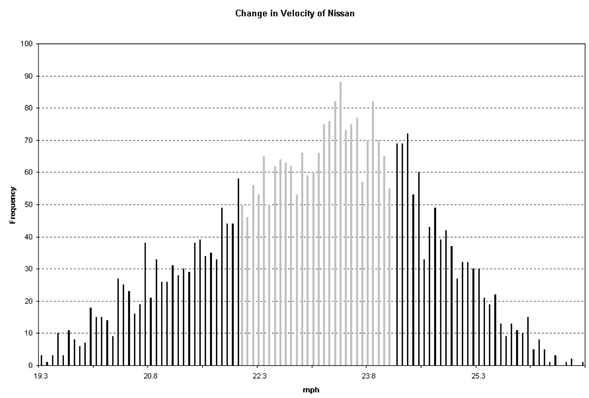

FIGURE 15

	Mean	Standard Deviation	51% Range
Chevy Change in Velocity	11.8	.7	11.3 to 12.4
Nissan Change in Velocity	23.1	1.4	22.0 to 24.1
Chevy Impact Speed	33.1	1.8	31.8 to 34.5
Nissan Impact Speed	4.8	2.0	3.3 to 6.2

TABLE 2

CONCLUSIONS AND RECOMMENDATIONS

MONTE CARLO SIMULATION

1. Monte Carlo simulation provides a statistical analysis technique to analyze the propagation of uncertainty in crash severity analysis. The results of a Monte Carlo simulation allow the reconstructionist to draw conclusions about the probable crash severity with an understanding of which values are truly most likely.

2. The Monte Carlo technique cannot establish the fidelity of the impact model in any particular case. Meaningful application of the Monte Carlo technique must come after there has already been a strong link established between the impact model and the accident that the model mimics. In other words, the assumptions of the model must fit well with the actual accident.

3. Poorly chosen ranges for the input parameters undermine the effectiveness and usefulness of the Monte Carlo simulation technique. The range for each parameter should be wide enough to encompass the full range of possible values.

4. Selecting the uniform distribution to describe variations in the independent variables represents the most conservative assumption since the value of the parameter is allowed to vary more widely than with any other distribution and will result in the widest variation in the dependent variable. The uniform distribution should be preferred for most independent parameters in crash severity analysis since data that would justify choosing any other distribution is typically unavailable.

5. A custom distribution can be constructed to represent variations in the coefficients of friction that recognizes that random variations in the friction values vary normally, but that also gives adequate consideration to uncertainty in the mean and standard deviation of that normal distribution.

6. Certain combinations of values of the independent parameters may produce unrealistic results, and the output of the Monte Carlo simulation should be filtered. Ranges formulated for the coefficients of restitution provide one criterion by which to judge the reasonableness of the result of each combination and should be used to rule out certain combinations. However, the range set on normal and tangential restitution values should not be overly restrictive, giving adequate consideration to the uncertainty associated with restitution values.

THE IMPACT MODEL

1. The goal of this discussion has not been to defend the fidelity and superiority of any particular impact model, or even to defend the use of certain constraint parameters within the context of momentum impact models. The model utilized in this paper was developed to give the reader insight into issues surrounding the analysis of uncertainty in impact modeling and crash severity analysis.

2. The impact model developed makes use of coefficients of restitution in both the normal and tangential directions. These coefficients of restitution are allowed to vary between -1 and 1, with the implication that there is no common velocity requirement inherent in the model.

3. Parting with a common velocity requirement is advantageous since the impact model can handle sideswipe and break-through collisions.

UNCERTAINTY AND RESTITUTION

1. Restitution in vehicular collisions is a complex structural phenomenon where the non-homogeneity of the vehicle structure causes the restitution response to vary depending on what portions of the vehicle are engaged in the impact. Also, each vehicle combination will exhibit a unique collision response at any given impact configuration and severity.

2. Researchers have generally concluded that normal restitution values decrease as crash severity increases. However, the non-homogeneity of the vehicle structure introduces important exceptions to the trend.

3. Ishikawa has presented a series of staged collisions that seem to indicate that any relationship between crash severity and restitution breaks down unless collisions are first grouped by their impact configuration.

4. Restitution values show considerable variation from test to test. Ranges formulated for the coefficient of restitution should be wide enough to encompass this variation.

ACKNOWLEDGMENTS

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A copy of the Excel file used for the analysis in this paper will be provided upon request.